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Lunar laser ranging and fundamental astrometry

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The frame of reference to which a positional astronomer refers his observations is defined by the directions of the Earth's axis and of the equinox. This frame rotates relative to an inertial frame owing to the precessional motion.

Traditionally, the precession is determined from an analysis of stellar proper motions on the assumption that their average kinematic behaviour follows a very simple pattern of parallactic motion and galactic rotation. Recent work has attempted to determine precession by measuring motions of stars relative to extragalactic objects.

An alternative method, advocated by the late G. M. Clemence, is to measure the apparent absolute rotations of planetary orbits, but so far the results from this approach have not been entirely satisfactory.

In this paper the traditional method of setting up a fundamental reference frame and determining precession will be reviewed, and the possibility of using the very high precision of the lunar laser ranging measurements to determine the rotation of the lunar orbit will be investigated.

1. Introduction

The instantaneous reference frame, with respect to which observations of positions of celestial objects, at any epoch, are referred, is defined kinematically by the the directions of the Earth's axis and the line of equinoxes. Both these directions must be determined relative to the reference frame of the observing instrumental system, by means of observations which, in practice, extend over finite time intervals.

Because of the luni-solar precession of the Earth's axis and the slow rotation of the ecliptic, the 'fixed' stars appear to move rapidly relative to the instantaneous celestial reference frame; it is thus convenient to define a quasi-inertial frame by adopting numerical expressions to describe the precessional rotations. It is this conventional reference frame which forms the basis of fundamental star catalogues and of apparent ephemerides of objects in the Solar System.

The objectives of fundamental astrometry are therefore twofold:

(1) To construct and maintain catalogues of positions and proper motions of stars in order to render the conventional reference frame generally accessible to observation.

(2) To measure the departure of the conventional reference frame from a truly inertial system.

2. The instantaneous celestial reference frame

Traditionally the north pole of the instantaneous reference frame has been defined to coincide with the axis of rotation of the Earth. Recently, Atkinson (1973) has drawn attention to observational difficulties associated with this pole, and pointed out that the pole of figure would be more appropriate. However, all that is required is an axis whose direction can in principle be calculated from dynamical theory, as a function of the time, with sufficient accuracy for satisfactory interpolation over the time intervals covered by the observations used for determining its direction relative to the instrumental frame. In the case of a classical transit circle, circumpolar stars are observed at both upper and lower transits to fix the azimuth and colatitude; for the former, observations may extend over a few days, whereas for the latter, several years observations are generally combined after due allowance for variation of latitude.

3. The conventional reference frame

Having determined the direction of the pole relative to the instrumental system throughout the period covered by a series of observations, it is in principle possible to form a catalogue of star positions (and variations) in which the declinations are fundamental, but the right ascensions are relative to an arbitrary zero point. It is thus necessary to determine the direction of the line of equinoxes, as defined kinematically, relative to the catalogue right ascensions.

Table 1. Standard errors of single observations

(Cooke, T.C., Herstmonceux.)

		$15\epsilon_\alpha \cos\delta$	ϵ_δ
Moon:	single limb	±0.87″	±1.35″
	Mösting A	0.70	0.98
Sun:	two limbs	0.80	0.62
stars		0.29	0.43

This direction has been derived most frequently from meridian observations of the Sun; in this case the determination is virtually independent of any dynamical theory of the Sun's motion. On the other hand, observationally the Sun is not an ideal object for the purpose. The observations of the limbs of the solar disk are liable to errors of a different nature from those affecting stellar objects, but more important, the thermal states of the instrument, and the atmosphere, are quite different at the time of the transit of the Sun, compared with those at night when most of the stars are observed.

Observations of other bodies in the Solar System can also be used to define the ecliptic plane, and hence the line of equinoxes, but in these cases the determination depends much more on the dynamical theories of their motions. Attempts have been made to use minor planets, but in practice it is difficult to separate the equinox correction from corrections to the orbital elements of the planets and the Earth, unless observations extend over a large arc of the orbits, which is seldom the case (Jackson 1968; Fricke 1972).

Lunar occultations have been used for determining the equinox (Newcomb 1912; Spencer Jones 1929; Morrison & Sadler 1969). Although the Moon is generally observed during the night, the difficulties for the meridian observer which are associated with the finite disk are more serious than for the Sun because of the irregularity of the limb and the fact that the position of the centre has to be inferred from single limb observations in each coordinate. These difficulties are much reduced if one observes a surface feature, such as the crater Mösting A, although errors depending on phase may still persist. This particular crater has been observed regularly since 1905 on the meridian programmes at Greenwich and Herstmonceux. It was first observed for a special programme, jointly with the Cape Observatory for determining the lunar parallax (Crommelin 1911). The standard errors of single determinations of the centres, of the Moon, the Sun and stars, deduced from current visual observations on the Cooke Transit Circle at Herstmonceux are summarized in table 1. Substantially higher accuracy on stars can be achieved with the use of photographic and photoelectric observing techniques (Høg 1974).

We see that the observations of Mösting A are substantially better than those of the lunar limb even though limb corrections have been applied, and could well be used for determining the equinox, provided that the lunar ephemeris were sufficiently precise. Attempts were made to use observations made at Greenwich from 1905 to 1939 for this purpose, but these led to inconclusive results and were never published.

With the great improvement in the lunar ephemeris which will be possible from the results of lunar laser ranging, it would not be necessary to allow for orbital and librational corrections in any future analysis of meridian observations, provided that the selenographic coordinates of the crater were well tied in to the system defined by the retro-reflectors. Indeed, any observable surface feature on the Moon could be used as an absolute standard of right ascension and declination as a function of atomic time T.A.I., which could be used to control the observed star positions in the zodiacal belt.

4. Absolute rotation of the conventional reference frame

The departure of the conventional reference frame from an inertial frame must be measured by observing some phenomenon which can be predicted absolutely by dynamical theory. As Newcomb (1897) has remarked, the actual observable effects on planetary orbits which would arise on account of an erroneous value of the luni-solar precession are very small. Clemence (1966) showed that the precession could indeed be determined from observations of the secular variations of the rotational elements of the orbits of Mercury, Venus and the Earth, but the precision attainable was limited at that time by uncertainties in planetary masses, particularly those of Mercury and Mars (see also Wayman 1966).

Much higher formal precision in measuring the rotation of the reference frame can be achieved by analysing proper motions of stars. If $\boldsymbol{r}$ is the position vector of a star we may express its absolute proper motion by

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \wedge \boldsymbol{r}, \tag{1}$$

where $\partial\boldsymbol{r}/\partial t$ represents the apparent proper motion measured relative to the conventional reference frame (fundamental catalogue) and $\boldsymbol{\omega}$ is a small vector representing the rotation of the conventional frame relative to an inertial frame.

In the absence of a detailed dynamical theory of the motions of stars in the Galaxy, equation (1) has been used for determining a number of parameters describing a simple model for $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ as well as components of $\boldsymbol{\omega}$, from measured proper motions $\partial\boldsymbol{r}/\partial t$. It is customary to include in the model, (i) Oort's constants A, B characterizing the galactic shear and differential rotation, parallel to the galactic plane, and (ii) the three components of the vector secular parallax of the centroid of stars used in the analysis. Although, within a distance of a few hundred parsecs, (i) is independent of distance, the parallactic motion (ii) depends in principle on a knowledge of the relative distances of stars. In spite of the obvious limitations of such a model, such as the neglect of differential motion depending on distance from the galactic plane, and also the effects of other possible non-random motions, this classical approach yields surprisingly good results.

Let $\boldsymbol{k}$, $\boldsymbol{h}$ be unit vectors directed towards the poles of the ecliptic and mean equator respectively at some standard epoch. The corresponding mean equinox is then in the direction

$$\boldsymbol{i} = \boldsymbol{h} \wedge \boldsymbol{k} \operatorname{cosec} \epsilon,$$

where ϵ is the mean obliquity of the ecliptic. We can write, formally,

$$\boldsymbol{\omega} = -\Delta\dot{\Psi}\boldsymbol{k} + \Delta\dot{\alpha}\boldsymbol{h} - \Delta\dot{e}\boldsymbol{i}, \tag{2}$$

where $\Delta\dot{\Psi}$ is a correction to the conventional value of the luni-solar precession, $\Delta\dot{\alpha}$ is a correction to proper motions in right ascension and $\Delta\dot{\epsilon}$ is a rotation of the equator about the line of equinoxes. The component $\Delta\dot{\alpha}$ is generally included in any analysis in order to account for a spurious 'motion of the equinox' which has arisen through variation of the systematic errors affecting determination of the equinox corrections over the past two centuries.

Clearly, only three independent components of rotation can be determined. Since galactic rotation (Oort's constant B) is included, it is not possible to derive three components of $\boldsymbol{\omega}$. It is customary to ignore $\Delta\dot{\epsilon}$, but it so happens that the pole of the Galaxy is well inclined to both $\boldsymbol{k}$ and $\boldsymbol{h}$ and hence $\Delta\dot{\epsilon}$, if it exists, will vitiate the determination of B (Aoki 1967; Fricke 1972). However, the theoretical motion of the equator is well understood and there is no reason to suppose that $\Delta\dot{\epsilon} \neq 0$.

Table 2. Observed rotation of conventional reference frame

(Unit: 0.01″/century.)

	Lick	Pulkovo	FK4 Stars
Δn	$+32 \pm 10$	$+41 \pm 12$	$+43 \pm 6$
Δk	-79 ± 10	$+43 \pm 12$	-22 ± 9
$\Delta\dot{\epsilon}$	$+17 \pm 11$		

Fricke (1968) has carried out an extensive analysis of the proper motions of 512 distant stars on the FK4 system. His results for the precessional rotations $\Delta n = \Delta\dot{\Psi}\sin\epsilon$, $\Delta k = \Delta\dot{\Psi}\cos\epsilon - \Delta\dot{\alpha}$, are summarized above in the third column of table 2.

Results are now becoming available from two observational programmes which are designed to give absolute proper motions of stars relative to extra-galactic objects. By comparing these absolute motions, with the apparent motions measured relative to the conventional reference frame (FK4) it is possible to insert $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ for each star into (1), and hence to determine the three components of $\boldsymbol{\omega}$ directly.

Vasilevskis & Klemola (1971) have given the results of a pilot programme for the Lick proper motion survey. They carried out solutions both with and without a $\Delta\dot{\epsilon}$ term; since this is orthogonal to Δn, Δk the results for the two latter components were essentially the same in both solutions. The weighted average result from analyses of two groups of stars (mean $m_{\mathrm{pg}} = 11.0$ and 10.1) measured on the same plates relative to the same galaxies are also given in the first column of table 2. The standard errors quoted are the present authors' estimates based on the assumption that the accuracy of measurement of a star image is about twice that of a galaxy image (Vasilevskis 1957).

The second column of table 2 gives the results of the Pulkovo programme of measuring absolute proper motions relative to galaxies (Fatchikhin 1970).

We see that there is general agreement, within the standard errors, for Δn and hence for the luni-solar precession $\Delta\dot{\Psi}$, but that there is as yet an unexplained divergence between values of Δk. Even so, the luni-solar precession must still be uncertain by at least 0.1″/century, which corresponds to a velocity of 5 km/s at a distance of a kiloparsec. This is larger than the accidental errors of good radial velocity observations and is half the generally accepted value of Oort's B constant.

5. Precession from lunar laser ranging

The high precision which is attainable in observations of the lunar distance may well make it possible to determine the luni-solar precession with an accuracy which cannot be achieved by current techniques of positional astronomy.

In order to examine this possibility, we have carried out a trial solution of fictitious observations which are supposed to extend over a full revolution of the Moon's node (235 lunations) with an average of five observations per lunation centred around full moon and extending over $180° \pm 140°$ in elongation from the Sun. The observing station is assumed to be in geocentric latitude $\phi' = 30°$.

The basic assumption is that the lunar orbit, including the motions of the node and perigee, is known absolutely except for the arbitrary constants of the theory, and we seek to determine the apparent motion of the node from observations.

Each observation gives an equation of condition relating

(i) the constants of the lunar and solar orbits,
(ii) the orientation of the instantaneous celestial reference frame with respect to the ecliptic,
(iii) the coordinates of the observer relative to the instantaneous celestial reference frame,
(iv) the orientation of the lunar reference frame relative to the ecliptic,
(v) the coordinates of the retro-reflectors relative to the lunar reference frame.

Since we are primarily interested only in eliminating terms arising from errors in the lunar ephemeris, rather than determining them absolutely, we have ignored (iv) and (v) on the assumption that these will correlate with (i) rather than (ii) and (iii).

The coordinates of the observer relative to the conventional reference frame (star catalogue system) can be determined by direct astronomical observation at any epoch, or inferred from observations made at the numerous time and latitude stations; we have therefore assumed these to be known *a priori*. However, the conventional reference frame may be rotated about the pole relative to the instantaneous celestial frame on account of a zero point error in the right ascension system of the catalogue, and we have allowed for this.

Let $\boldsymbol{S}$ represent the geocentric vector to the centre of the Moon and $\boldsymbol{R}$ that to the observer. If $\boldsymbol{k}$, $\boldsymbol{n}$ are unit vectors directed towards the pole of the ecliptic and true equator respectively, and $\boldsymbol{l}$ is toward the true equinox, we have

$$\boldsymbol{S} = \sigma^{-1}[\cos\beta\cos\lambda\boldsymbol{l} + \cos\beta\sin\lambda\boldsymbol{k}\wedge\boldsymbol{l} + \sin\beta\boldsymbol{k}], \tag{3}$$

where λ, β, σ are the lunar ecliptic longitude, latitude and sine parallax respectively. Also we can write

$$\boldsymbol{R} = \boldsymbol{R}_0 + \Delta\alpha\boldsymbol{n}\wedge\boldsymbol{R}_0, \tag{4}$$

where $$\boldsymbol{R}_0 = \rho[\cos\phi'\cos\tau\boldsymbol{l} + \cos\phi'\sin\tau\boldsymbol{n}\wedge\boldsymbol{l} + \sin\phi'\boldsymbol{n}],$$

$\Delta\alpha$ is the correction required to be added to the right ascension of the star catalogue system, τ is the local sidereal time relative to that system, and ρ, ϕ' are the geocentric distance and latitude of the observer.

The range equation can be written

$$\Delta d = \frac{(\boldsymbol{S}-\boldsymbol{R}_0)}{|\boldsymbol{S}-\boldsymbol{R}_0|}\cdot(\Delta\boldsymbol{S}-\Delta\boldsymbol{R}), \tag{5}$$

where d is the observed distance.

Since we are assuming that the components of $\boldsymbol{R}$ relative to the conventional frame are known, the only contributions to (5) from $\Delta\boldsymbol{R}$ are from corrections to the luni-solar precession, the right ascension zero point and the obliquity of the ecliptic. All three of these can be functions of time, and the two latter may also have a constant term as well; a constant rotation in longitude is of course indistinguishable from a correction to the longitude of the lunar node. Hence we write $\Delta\boldsymbol{R}$ in the form

$$\Delta\boldsymbol{R} = \{-t\Delta\dot{\Psi}\boldsymbol{k} + (\Delta\alpha + t\Delta\dot{\alpha})\,\boldsymbol{n} - (\Delta\epsilon + t\Delta\dot{\epsilon})\,\boldsymbol{l}\} \wedge \boldsymbol{R}. \tag{6}$$

Although, as we remarked above, we should expect $\Delta\dot{\epsilon}$ to be zero, we retain it in the solution in case it is not zero, in view of the importance of the influence of such a rotation on the determination of Oort's B constant.

We write

$$\Delta\boldsymbol{S} = -\frac{\Delta\sigma}{\sigma}\boldsymbol{S} + (\boldsymbol{W}_1 + \boldsymbol{W}_2) \wedge \boldsymbol{S}, \tag{7}$$

where $\boldsymbol{W}_1$ contains only corrections to the dynamical elements of the orbits of the Moon and Earth, and $\boldsymbol{W}_2$ contains corrections to the rotational elements of the lunar orbit.

In the notation of Brown's *Tables of the motion of the Moon* (1919) we include corrections to the following dynamical elements:

l, l' mean anomalies of Moon and Sun;
D mean elongation of Moon from Sun;
e, e' eccentricities of lunar and solar orbits;
a unperturbed mean distance of Moon from Earth (at zero epoch);
$\alpha_1 = \dfrac{E-M}{E+M}\dfrac{a}{a'}$ constant of parallactic terms, in which E, M are masses of Earth and Moon, and a' is the semi-major axis of the Earth's orbit;
n, n' sidereal mean motions of Moon and Sun;
$\dot{n}$ sidereal secular acceleration of the Moon.

If $\Delta\lambda$, $\Delta\beta$ denote corrections to the ecliptic longitude and latitude arising only from corrections to these dynamical elements, we have

$$\boldsymbol{W}_1 = \Delta\beta\sin\lambda\boldsymbol{l} - \Delta\beta\cos\lambda\boldsymbol{k}\wedge\boldsymbol{l} + \Delta\lambda\boldsymbol{k}. \tag{8}$$

Further, if i, Ω, $\tilde{\omega}$ are the inclination, longitude of node and argument of perigee of the lunar orbit, then

$$\boldsymbol{W}_2 = (\Delta i\cos\Omega + \Delta\tilde{\omega}\sin i\sin\Omega)\,\boldsymbol{l} + (\Delta i\sin\Omega - \Delta\tilde{\omega}\sin i\cos\Omega)\,\boldsymbol{k}\wedge\boldsymbol{l} + (\Delta\Omega + \Delta\tilde{\omega}\cos i)\,\boldsymbol{k}. \tag{9}$$

It should be noted, that, since the Moon's argument of latitude, F, is equivalent to $l+\tilde{\omega}$, we must put $\Delta F = \Delta l$ in $\boldsymbol{W}_1$ since $\Delta\tilde{\omega}$ is included in $\boldsymbol{W}_2$.

In forming the differential coefficients of λ, β, σ with respect to the various dynamical elements, numerical values of the coefficients of the significant terms were taken from Brown's tables. The terms which were actually included are represented by the following expressions, which are correct to second order of small quantities:

$$\lambda = l + \tilde{\omega} + \Omega + 2e\sin l + \tfrac{5}{4}e^2\sin 2l - \gamma^2\sin 2F - \tfrac{15}{4}me\sin(l-2D) - 3me'\sin l' - \tfrac{15}{8}m\alpha_1\sin D + \tfrac{11}{8}m^2\sin 2D, \tag{10}$$

$$\beta = 2\gamma\sin F + 2e\gamma\sin(l-F) + 2e\gamma\sin(l+F) - \tfrac{3}{4}m\gamma\sin(F-2D), \tag{11}$$

$$a\sigma = 1 + \tfrac{1}{6}m^2 + e\cos l + e^2\cos 2l + \tfrac{15}{8}me\cos(l-2D) - \tfrac{15}{16}m\alpha_1\cos D + m^2\cos 2D, \tag{12}$$

in which $\gamma = \sin\frac{1}{2}i$ and $m = n'/(n-n')$. Even though corrections to the mean motions n, n' have been included when they arise from corrections to the arguments, the ratio m has been supposed known absolutely and not subject to correction.

From Kepler's third law the unperturbed mean distance, a, is related to n by

$$n^2a^3 = E+M. \tag{13}$$

Therefore, since we are allowing for variation of both n and $\dot{n}$ we must write

$$\begin{aligned}\frac{\Delta a}{a} &= \frac{1}{3}\frac{\Delta(E+M)}{E+M} - \frac{2}{3}\frac{(\Delta n + t\Delta\dot{n})}{n}\\ &= \left(\frac{\Delta a}{a}\right)_0 - \frac{2t}{3n}\Delta\dot{n},\end{aligned} \tag{14}$$

where t is the time. Accordingly

$$\frac{\Delta\sigma}{\sigma} = -\left(\frac{\Delta a}{a}\right)_0 + \frac{2t}{3n}\Delta\dot{n} + \frac{1}{a\sigma}\Delta(a\sigma), \tag{15}$$

where $\Delta(a\sigma)$ represents the differential coefficients of $a\sigma$ with respect to all the dynamical elements, including $\Delta\dot{n}$.

6. Discussion of results

Solutions were carried out at intervals of a year, over the full 18-year period. The formal standard errors of the five unknowns associated with the rotation of the equatorial reference frame are listed in table 3 for two solutions: (i) 575 observations extending over 9 years and (ii) 1151 observations extending over 18 years. The standard error of unit weight for a single observation of distance has been taken as ± 10 cm or $\pm 3.23 \times 10^{-3}$ arcsecond.

Table 3. Standard errors of rotation components and centennial variations

	(i)	(ii)
$\Delta\epsilon$	$\pm 0.0018''$	$\pm 0.0004''$
$\Delta\alpha$	$0.0042''$	$0.0010''$
$\Delta\dot{\epsilon}$	$0.0456''$	$0.0062''$
$\Delta\dot{\Psi}\cos\epsilon - \Delta\dot{\alpha}$	$0.0182''$	$0.0033''$
$\Delta\dot{\Psi}\sin\epsilon$	$0.0457''$	$0.0066''$

The zero epoch for all solutions was fixed near the mid point of the 18-year period; there are thus significant correlations still affecting the 9-year solution, particularly between the constant and secular parts of the rotation components. Even 9-year solutions for the secular components give a formal precision which is considerably better than that achieved by classical methods (see table 2).

It should be remembered that a basic assumption in these calculations has been that the secular variation of the longitude of the Moon's node, the argument of perigee and the inclination are known absolutely from theory. Furthermore, inclusion of terms representing lunar rotation and reflector coordinates may well weaken the solution for the rotation components of the reference frame. Therefore, while these results are encouraging, it is by no means evident that they will be achieved in practice, and it would be unwise at the present time to relax the efforts now being made to determine the inertial reference frame by classical astrometric techniques.

The final form of presentation of the results in this paper have been influenced by discussions with other participants at the meeting, particularly Dr J. G. Williams and Dr D. E. Smith, to whom the authors are most grateful.

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